BRST OPERATOR FOR QUANTUM LIE ALGEBRAS: RELATION TO BAR COMPLEX

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Abstract

Quantum Lie algebras (an important class of quadratic algebras arising in the Woronowicz calculus on quantum groups) are generalizations of Lie (super) algebras. Many notions from the theory of Lie (super)algebras admit "quantum" generalizations. In particular, there is a BRST operator Q ($Q^2 = 0$) which generates the differential in the Woronowicz theory and gives information about (co)homologies of quantum Lie algebras. In our previous papers a recurrence relation for the operator Q for quantum Lie algebras was given and solved. Here we consider the bar complex for q-Lie algebras and its subcomplex of q-antisymmetric chains. We establish a chain map (which is an isomorphism) of the standard complex for a q-Lie algebra to the subcomplex of the antisymmetric chains. The construction requires a set of nontrivial identities in the group algebra of the braid group. We discuss also a generalization of the standard complex to the case when a q-Lie algebra is equipped with a grading operator.

1 Introduction

The Woronowicz calculus [1] associates an algebra of exterior forms Γ and an enveloping algebra $\mathcal{U}$ of the left invariant vector fields on $\mathcal{A}$ to a Hopf algebra $\mathcal{A}$. The algebra $\mathcal{U}$ is called a quantum Lie algebra (q-Lie algebra for short). It is defined by relations $\chi_i\,\chi_j - \sigma_{ij}^{km}\,\chi_k\,\chi_m = C_{ij}^k\,\chi_k$, where $\{\chi_i\}$ is a set of

generators; "structure constants" σ_{ij}^{km} and C_{ij}^{k} obey certain constraints, see Section 2. A q-Lie algebra is a non-homogeneous quadratic algebra. The general theory of quadratic algebras has been considered in a number of papers (see e.g. [2, 3, 4]). The case of q-Lie algebras is quite particular. An analog of the de Rham complex for $\mathcal{U}$ has been constructed in [1]. Mimicking the classical theory of Lie algebras, quantum analogs of the standard complex and BRST differential (with expected properties) have been introduced in [5, 6] (for a review of the general BRST-theory see [7]). In this paper we continue the investigation of q-Lie algebras along the lines of the theory of Lie algebras. Namely, (see e.g. [8]) we map the standard complex of the q-Lie algebra into the subspace of q-antisymmetric chains of the bar complex for $\mathcal{U}$. It turns out that the image of this map, just like in the case Lie algebras, is a subcomplex in the bar complex for $\mathcal{U}$. Moreover, the quantum BRST differential is the restriction of the bar differential b to the subspace of the q-antisymmetric chains.

The paper is organized as follows. In Section 2 we recall the definition of q-Lie algebras. In Section 3 we introduce the exterior extensions of q-Lie algebra (an algebra of exterior forms Γ and inner derivatives Γ^*) due to Woronowicz (our definition of the basis of differential forms is slightly different from the Woronowicz definition, see [5]) and develop the tools needed for the main constructions of Sections 4 and 5. Section 4 contains an explicit construction of the BRST operator of a q-Lie algebra. The presentation in the Sections 2-4 follows [12].

In Section 5 we discuss the bar complex $(C_n(\mathcal{U}), b)$ for a q-Lie algebra, its subcomplex of q-antisymmetric chains $(C_n(\mathcal{U}, \wedge\mathcal{L}), b)$ and a chain map of the standard complex for q-Lie algebra to $(C_n(\mathcal{U}, \wedge\mathcal{L}), b)$. Nontrivial identities in braid group algebra are used for proving the main Propositions of Sections 4 and 5.

Section 6 is a generalization of the constructions above to the situation when a q-Lie algebra is equipped with a grading operator. An example: a Lie super-algebra with grading given by parity. It is known that there are two choices of commutation relations between bosonic and fermionic ghosts. We explain this phenomenon in a general framework of a q-Lie algebra with a grading operator.

Section 7 summarizes the results of the paper.

2 Definitions and notation

The data defining a quantum Lie algebra with N generators can be conveniently encoded into the following $(N+1)^2 \times (N+1)^2$ matrix [9]:

$$
\begin{aligned}
R^{CD}_{AB} &= \delta^i_A\, \delta^j_B\, \sigma^{kl}_{ij}\, \delta^C_k\, \delta^D_l + \delta^i_A\, \delta^j_B\, C^k_{ij}\, \delta^C_0\, \delta^D_k + \delta^i_A \delta^D_i\, \delta^0_B\, \delta^C_0 + \delta^C_B\, \delta^0_A\, \delta^D_0 \\
&\equiv \delta^{\langle 1}_A\, \delta^{\langle 2}_B\, \sigma_{12}\, \delta^C_{1\rangle}\, \delta^D_{2\rangle} + \delta^{\langle 1}_A\, \delta^{\langle 2}_B\, C^{\langle 2}_{12\rangle}\, \delta^C_0\, \delta^D_{2\rangle} + \delta^{\langle 1}_A \delta^D_{1\rangle}\, \delta^0_B\, \delta^C_0 + \delta^C_B\, \delta^0_A\, \delta^D_0 \ .
\end{aligned}
\tag{1}
$$

Here $1, 2, \ldots$ denote copies of an N-dimensional vector spaces V_N; $\langle 1, \langle 2, \ldots$ (resp., $1\rangle, 2\rangle, \ldots$) are the corresponding outcoming and incoming vectors; small letters $i, j, k, l, \ldots = 1, 2, \ldots, N$ denote indices of the vector space V_N; capital letters $A, B, \ldots = 0, 1, \ldots, N$ denote indices of an $(N+1)$-dimensional space V_{N+1} (V_N is a subspace in V_{N+1}).

Eqs. (1) are equivalent to

$$
R^{ij}_{kl} = \sigma^{ij}_{kl} \ , \quad R^{0j}_{kl} = C^j_{kl} \ , \quad R^{0A}_{B0} = R^{A0}_{0B} = \delta^A_B \tag{2}
$$

and the other components of R vanish. We assume that the matrix $\sigma \in End(V_N \otimes V_N)$ has an eigenvalue 1.

Let $L_A = \{\chi_0, \chi_i\}$ ($i = 1, \ldots, N$) be a "quantum vector":

$$
L_A = \delta^{\langle 1}_A \chi_{1\rangle} + \delta^0_A\, \chi_0 = \begin{pmatrix} \chi_0 \\ \chi_i \end{pmatrix} \tag{3}
$$

for the matrix R:

$$
R^{CD}_{AB}\, L_C\, L_D = L_A\, L_B \ . \tag{4}
$$

These relations are equivalent to

$$
[\chi_0,\, \chi_i] = 0 \quad \text{and} \quad (1 - \sigma_{12})\, \chi_{1\rangle}\, \chi_{2\rangle} = C^{\langle 1}_{12\rangle}\, \chi_0\, \chi_{1\rangle} \ .
$$

One can rescale the elements, $\chi_i \to \chi_0\, \chi_i$. The algebra with rescaled generators χ_i ($i = 1, 2, \ldots, N$) subject to relations

$$
(1 - \sigma_{12})\, \chi_{1\rangle}\, \chi_{2\rangle} = C^{\langle 1}_{12\rangle}\, \chi_{1\rangle} \ , \tag{5}
$$

is called a *quantum Lie algebra* if the matrix R satisfies the Yang-Baxter equation,

$$
R^{CD}_{AB}\, R^{ES}_{MC}\, R^{FN}_{SD} = R^{CS}_{MA}\, R^{DN}_{SB}\, R^{EF}_{CD} \quad \text{or} \quad R_{\underline{23}}\, R_{\underline{12}}\, R_{\underline{23}} = R_{\underline{12}}\, R_{\underline{23}}\, R_{\underline{12}} \tag{6}
$$

(here $\underline{1},\underline{2}$ or $\underline{2},\underline{3}$ denote copies of the vector spaces V_{N+1} on which R- matrices act nontrivially). The Yang-Baxter equation for R imposes the following conditions for the structure constants of the quantum Lie algebra (5):

$$\sigma_1\,\sigma_2\,\sigma_1 = \sigma_2\,\sigma_1\,\sigma_2 \ , \tag{7}$$

$$C_1\,\delta_3\,C_1 = \sigma_2\,C_1\,\delta_3\,C_1 + C_2\,C_1 \ , \tag{8}$$

$$C_1\,\delta_3\,\sigma_1 = \sigma_2\,\sigma_1\,C_2 \ , \tag{9}$$

$$(\sigma_2\,C_1\,\delta_3 + C_2)\,\sigma_1 = \sigma_1\,(\sigma_2\,C_1\,\delta_3 + C_2) \ . \tag{10}$$

Here we have used a concise notation $\sigma_n := \sigma_{n\,n+1}$, $C_n := C^{\langle n}_{n,n+1\rangle}$, $\delta_n := \delta^{\langle n-1}_{\ \ n\rangle}$.

3 Exterior extensions for quantum Lie algebra

To define the exterior extension of the quantum Lie algebra (5), one introduces quantum wedge algebras with generators γ_i and Ω^i $(i = 1, \dots N)$:

$$\Omega^{\langle n} \wedge \ldots \wedge \Omega^{\langle 2} \wedge \Omega^{\langle 1} = \Omega^{\langle n} \otimes \ldots \otimes \Omega^{\langle 2} \otimes \Omega^{\langle 1}\, A_{1\to n} \ , \tag{11}$$

$$\gamma_{1\rangle} \wedge \gamma_{2\rangle} \ldots \wedge \gamma_{n\rangle} = A_{1\to n}\, \gamma_{1\rangle} \otimes \gamma_{2\rangle} \ldots \otimes \gamma_{n\rangle} \ . \tag{12}$$

The cross-commutation relations are

$$\gamma_{2\rangle}\, \Omega^{\langle 2} = -\Omega^{\langle 1}\, \sigma_{12}^{-1}\, \gamma_{1\rangle} + I_2 \ . \tag{13}$$

In eqs. (11) and (12) we have used operators $A_{1\to n}$, quantum analogues of the antisymmetrizers. They can be defined inductively (in fact they make sense already for the group algebra of the braid group B_{M+1} with generators σ_i $(i = 1, \dots, M)$):

$$A_{1\to n} \equiv f_{1\to n}\, A_{1\to n-1} \quad \text{or} \quad A_{1\to n} \equiv \overline{f}_{1\to n}\, A_{2\to n} \ , \tag{14}$$

where $A_{1\to 1} = 1$, $f_{k\to k} = \overline{f}_{k\to k} = 1$ and

$$f_{k\to m} = 1 - f_{k\to m-1}\,\sigma_{m-1} \quad \text{or} \quad f_{k\to m} = f_{k+1\to m} + (-1)^{m-k}\,\sigma_k \cdots \sigma_{m-1} \ ,$$

$$\overline{f}_{k\to m} = 1 - \overline{f}_{k+1\to m}\,\sigma_k \quad \text{or} \quad \overline{f}_{k\to m} = \overline{f}_{k\to m-1} + (-1)^{m-k}\,\sigma_{m-1} \cdots \sigma_{k+1}\,\sigma_k \tag{15}$$

for $(k < m \leq n)$. Explicitly:

$$f_{k\to n} \equiv f^{(\sigma)}_{k\to n} = 1 - \sigma_{n-1} + \sigma_{n-2}\,\sigma_{n-1} - \ldots + (-1)^{n-k}\,\sigma_k\,\sigma_{k+1}\cdots\sigma_{n-1}\ ,$$

$$\overline{f}_{k\to n} \equiv \overline{f}^{(\sigma)}_{k\to n} = 1 - \sigma_k + \sigma_{k+1}\,\sigma_k - \ldots + (-1)^{n-k}\,\sigma_{n-1}\cdots\sigma_{k+1}\,\sigma_k\ . \tag{16}$$

If the sequence of operators $A_{1\to n}$ terminates at the step $n = h+1$ ($A_{1\to n} = 0$ $\forall n > h$) for some matrix representation ρ of B_{M+1} then the number h is called the height of the matrix $\rho(\sigma_{12})$.

The elements γ_i and Ω^i are quantum analogues of ghost variables. The choice of the sign and the appearance of the braid matrix in (13) follows the conventional case of Lie (super)algebras. The commutation relations of χ_i with γ_j and Ω^k are

$$\chi_{2\rangle}\,\Omega^{\langle 2} = \Omega^{\langle 1}\,\left(\sigma_{12}\,\chi_{1\rangle} + C^{\langle 2}_{12\rangle}\right)\ ,\quad \gamma_{1\rangle}\,\chi_{2\rangle} = \sigma_{12}\,\chi_{1\rangle}\,\gamma_{2\rangle} + C^{\langle 1}_{12\rangle}\,\gamma_{1\rangle}\ . \tag{17}$$

The algebra (5), (11) – (13) and (17) is the exterior extension of the quantum Lie algebra and (see [5]) this algebra gives rise to the Cartan differential calculus on quantum groups in the Woronowicz theory [1].

Remark. Let $\hat{\sigma}_i$ $(i = 1,\ldots M)$ be generators of the braid group B_{M+1}:

$$\hat{\sigma}_i\,\hat{\sigma}_{i+1}\,\hat{\sigma}_i = \hat{\sigma}_{i+1}\,\hat{\sigma}_i\,\hat{\sigma}_{i+1}\ ,\quad [\hat{\sigma}_i,\hat{\sigma}_j] = 0 \text{ for } |i-j| > 1\ . \tag{18}$$

The important properties of the elements $f^{(\hat{\sigma})}_{k\to n}$ and $\overline{f}^{(\hat{\sigma})}_{k\to n}$ (16) of B_{M+1} are:

$$f^{(\hat{\sigma})}_{1\to n} f^{(\hat{\sigma})}_{1\to n-1}\cdots f^{(\hat{\sigma})}_{1\to m} = x^{(n-m+1)}_n\,A_{m\to n}\ ,\quad (n \geq m \geq 1) \tag{19}$$

$$\overline{f}^{(\hat{\sigma})}_{1\to n}\overline{f}^{(\hat{\sigma})}_{2\to n}\cdots\overline{f}^{(\hat{\sigma})}_{m\to n} = x^{(n-m)}_n\,A_{1\to m}\ ,\quad (n \geq m \geq 1) \tag{20}$$

where $x^{(0)}_n = 1$, $x^{(1)}_m = f^{(\hat{\sigma})}_{1\to m}$, $x^{(n)}_n = 1$, $x^{(n-1)}_n = \overline{f}^{(\hat{\sigma})}_{1\to n}$ and $x^{(2)}_{m+1}$ is given by

$$\begin{aligned} x^{(2)}_{m+1} = {} & f^{(\hat{\sigma})}_{1\to m} + f^{(\hat{\sigma})}_{1\to m-1}\,(\hat{\sigma}_m\,\hat{\sigma}_{m-1}) + f^{(\hat{\sigma})}_{1\to m-2}\,(\hat{\sigma}_{m-1}\,\hat{\sigma}_{m-2})\,(\hat{\sigma}_m\,\hat{\sigma}_{m-1}) \\ & + \ldots + f^{(\hat{\sigma})}_{1\to 2}\,(\hat{\sigma}_3\,\hat{\sigma}_2)\cdots(\hat{\sigma}_m\,\hat{\sigma}_{m-1}) + (\hat{\sigma}_2\,\hat{\sigma}_1)\cdots(\hat{\sigma}_m\,\hat{\sigma}_{m-1})\ . \end{aligned} \tag{21}$$

The identities (19), (20) are equivalent to the factorization formula

$$A_{1\to n} = x^{(n-m)}_n\,A_{1\to m}\,A_{m+1\to n} \tag{22}$$

The elements $x_n^{(m)}$ can be defined inductively using the recurrent relation

$$x_{n+1}^{(n-m+1)} = x_n^{(n-m)} - (-1)^{n-m}\, x_n^{(n-m+1)}\, \hat{\sigma}_n \dots \hat{\sigma}_m$$

and in particular we have

$$x_{n+1}^{(2)} = (f_{1\to n}^{(\hat{\sigma})} + x_n^{(2)}\, \hat{\sigma}_n\, \hat{\sigma}_{n-1}) \ , \quad (n \geq 2) \ , \tag{23}$$

which gives (21). Then, we have $x_{n+1}^{(3)} = x_n^{(2)} - x_n^{(3)}\, \hat{\sigma}_n\, \hat{\sigma}_{n-1}\, \hat{\sigma}_{n-2}$ for $(n \geq 3)$, etc.

Note that the elements $x_n^{(m)}$ are alternating sums over the braid group elements which can be considered as quantum analogs of $(m,\, n-m)$ shuffles (the shuffles are obtained by projection $\hat{\sigma}_i \to -s_i$, where s_i are generators of symmetric group S_{M+1}). From this point of view the wedge products (11), (12) are related to the quantum shuffle products (about quantum shuffles and corresponding products see [10], [1]). The associativity of these products is provided by the identities: $x_n^{(n-m)}\, x_m^{(m-k)} = x_n^{(n-k)}\, (T_k\, x_{n-k}^{(n-m)}\, T_k^{-1})$ for $(k < m < n)$, where $T_k\, \hat{\sigma}_i\, T_k^{-1} = \hat{\sigma}_{i+k}$.

4 BRST operator for quantum Lie algebra

In the paper [5] we have given a recurrence which determines the BRST operator Q for the quantum Lie algebra:

Proposition 1. *The BRST operator Q for the quantum Lie algebras (5), which satisfies the equation $Q^2 = 0$, should have the following form*

$$Q = \Omega^i\, \chi_i + \sum_{r=1}^{h-1} Q_{(r)} \ , \tag{24}$$

where h is the height of the braid matrix σ_{12}, the operators $Q_{(r)}$ are given by

$$Q_{(r)} = \Omega^{\langle r+1|} \otimes \Omega^{\langle r|} \otimes \dots \otimes \Omega^{\langle 1|}\, (AXA)_{|1\dots r+1\rangle}^{\langle 1\dots r|}\, \gamma_{|1\rangle} \otimes \dots \otimes \gamma_{|r\rangle} \tag{25}$$

and $(AXA)_{|1\dots r+1\rangle}^{\langle 1\dots r|} = A_{1\to r+1} (XA)_{|1\dots r+1\rangle}^{\langle 1\dots r|} = (AX)_{|1\dots r+1\rangle}^{\langle 1\dots r|} A_{1\to r}$ are tensors which obey the following recurrent relations

$$A_{1\to r+1}\, (XA)_{|1\dots r+1\rangle}^{\langle 1\dots r|} = A_{1\to r+1}\, \left((-1)^r\, \sigma_r \sigma_{r-1} \cdots \sigma_1 - \mathbf{1}\right)\, (XA)_{|2\dots r+1\rangle}^{\langle 2\dots r|} \tag{26}$$

with the initial condition $A_{12}\, X^{\langle 1|}_{|12\rangle} = -C^{\langle 1|}_{|12\rangle}$.

It should be stressed that the derivation of the recurrence (26) requires the consideration of the terms (in the equation $Q^2 = 0$) linear in χ_i only.

In [12] we presented the solution of eqn. (26) and defined coefficients $(A\, X\, A)^{\langle 1\ldots r|}_{|1\ldots r+1\rangle}$ explicitly. This was done with the help of an important identity in the group algebra of the braid group:

Lemma 1. *Let* $\hat\sigma_i$ $(i = 1, \ldots M)$ *be generators of the braid group* B_{M+1} *(18). Then the following identity in the group algebra of* B_{M+1} *holds:*

$$Y_{1\to r+1}\, \overline{f}^{(\hat\sigma)}_{1\to r} = \Big((-1)^{r+1}\, f^{(\hat\sigma)}_{1\to r+1}\, \hat\sigma_r \cdots \hat\sigma_1 + \overline{f}^{(\hat\sigma)}_{1\to r+1} \Big)\, Y_{2\to r+1}\ , \tag{27}$$

where $f^{(\hat\sigma)}_{1\to r}$ *and* $\overline{f}^{(\hat\sigma)}_{1\to r}$ *are defined in (16) and*

$$\begin{aligned} &Y_{k\to r+1} = (1 - \hat\sigma_r^2)\, (1 + \hat\sigma_{r-1}\, \hat\sigma_r^2) \cdots \Big(1 - (-)^{r-k}\, \hat\sigma_k\, \hat\sigma_{k+1} \ldots \hat\sigma_{r-1}\, \hat\sigma_r^2 \Big)\ , \\ &Y_{k\to k} = 1\ . \end{aligned} \tag{28}$$

Proof. With the help of eqs. (15), the identity (27) can be represented in the form

$$Y_{1\to r+1}\, \overline{f}^{(\hat\sigma)}_{1\to r} = F_{1\to r+1}\, Y_{2\to r+1}\ , \tag{29}$$

where

$$F_{1\to r+1} = \Big((-1)^r\, f^{(\hat\sigma)}_{1\to r}\, \hat\sigma_r^2 \cdots \hat\sigma_1 + \overline{f}^{(\hat\sigma)}_{1\to r} \Big)\ . \tag{30}$$

One can prove the identity (29) by induction (see [12], [13]). •

Remark. The operators $Y_{k\to r+1}$ (28) have another form:

$$Y_{k\to r+1} = (1 + \hat\sigma_r)\, (1 - \hat\sigma_{r-1}\, \hat\sigma_r) \ldots (1 + (-1)^{r-k}\, \hat\sigma_k\, \hat\sigma_{k+1} \ldots \hat\sigma_r)\, f^{(\hat\sigma)}_{k\to r+1}\ . \tag{31}$$

This is nothing but the quantum version of the Zagier's factorization identities [11] (which can be obtained from (31) by the projection $\hat\sigma_i \to q\, s_i$, where s_i are generators of the symmetric group, $s_i^2 = 1$). The proof of identities (31) is straightforward (see e.g. [12]).

We have the following result [12]:

Proposition 2. *The explicit solution of the recurrent relations (26) is given by the formula*

$$
\begin{aligned}
&(-1)^{r+1}\, (AXA)^{i_1\dots i_r}_{j_1\dots j_{r+1}} \\
&= \Big((1-R^2_{\underline{r}})\Big(1+R_{\underline{r-1}}R^2_{\underline{r}}\Big)\cdots\Big(1+(-)^r R_{\underline{1}}\dots R_{\underline{r-1}}R^2_{\underline{r}}\Big)\Big)^{k_1\dots k_r\,0}_{j_1\dots j_r j_{r+1}}\,(A_{1\to r})^{i_1\dots i_r}_{k_1\dots k_r}\ .
\end{aligned}
\tag{32}
$$

Here $\underline{1},\dots,\underline{r-1},\underline{r}$ label the copies of the vector space V_{N+1}; $R_{\underline{r}} := R_{\underline{r,r+1}}$ and $\underline{r}$, $\underline{r+1}$ are the numbers of vector spaces V_{N+1} where the R-matrix (1) acts nontrivially; $i_1,\dots,j_1\dots,k_1,\dots = 1,2,\dots N$ are the vector indices.

Proof. Using the definition of the antisymmetrizer (14) and the braid relation (7) we rewrite the right-hand side of (26) in the form

$$
\begin{aligned}
&A_{1\to r+1}\,((-1)^r\,\sigma_r\sigma_{r-1}\cdots\sigma_1-\mathbf{1})\,(XA)^{\langle 2\dots r|}_{|2\dots r+1\rangle} \\
&=\Big((-1)^r\,f_{1\to r+1}\,\sigma_r\sigma_{r-1}\cdots\sigma_1-\overline{f}_{1\to r+1}\Big)\,A_{2\to r+1}\,(XA)^{\langle 2\dots r|}_{|2\dots r+1\rangle}
\end{aligned}
\ .
$$

Thus, the expression (32) is a solution of the equation (26) if the following identity holds

$$
\Big(Y_{\underline{1\to r+1}}\Big)^{\langle 1\dots r\,0}_{1\dots r+1\rangle}\,\overline{f}_{1\to r}=\Big((-1)^{r+1}\,f_{1\to r+1}\,\sigma_r\cdots\sigma_1+\overline{f}_{1\to r+1}\Big)\Big(Y_{\underline{2\to r+1}}\Big)^{\langle 2\dots r\,0}_{2\dots r+1\rangle}\ ,
\tag{33}
$$

where

$$
Y_{\underline{k\to r+1}}=(1-R^2_{\underline{r}})\Big(1+R_{\underline{r-1}}\,R^2_{\underline{r}}\Big)\cdots\Big(1+(-)^r\,R_{\underline{k}}\,R_{\underline{k+1}}\dots R_{\underline{r-1}}\,R^2_{\underline{r}}\Big)\ .
$$

As indicated by the structure of indices, the vector space V_{N+1} is a direct sum of V_N and a one-dimensional vector space V_0. Let $P : V_{N+1}\to V_N$ and $1-P : V_{N+1}\to V_0$ be the corresponding projectors. Similarly, let $\bar{P}$ and $1-\bar{P}$ be the projectors associated to the dual space $\bar{V}_{N+1}$.

Define

$$
P_{1\to r} := P_1\dots P_r\ ,\ P^{1\to r\,0}=\bar{P}_1\dots\bar{P}_r(1-\bar{P})_{r+1}\ .
$$

These operators split out the components which we need.

Eqn. (33) can be rewritten now in the form

$$
\begin{aligned}
& P_{1\to r+1}\, Y_{\underline{1\to r+1}}\, \overline{f}^{(R)}_{\underline{1\to r}}\, P^{1\to r\,0} \\
& = P_{1\to r+1} \left((-1)^{r+1}\, f^{(R)}_{\underline{1\to r+1}}\, R_{\underline{r}} \cdots R_{\underline{1}} + \overline{f}^{(R)}_{\underline{1\to r+1}} \right) Y_{\underline{2\to r+1}}\, P^{1\to r\,0}
\end{aligned} \tag{34}
$$

(where $f^{(R)}_{\underline{1\to n}}$ and $\overline{f}^{(R)}_{\underline{1\to n}}$ are the $(N+1)$-dimensional analogues of the operators (16)) in view of relations

$$
\overline{f}^{(R)}_{\underline{1\to r}}\, P^{1\to r} = P^{1\to r}\, \overline{f}^{(\sigma)}_{1\to r}\ ,
$$

$$
\begin{aligned}
Y_{\underline{2\to r+1}}\, P^{1\to r\,0} \quad &= \quad P^{1\to r+1} \left(P_{2\to r+1}\, Y_{\underline{2\to r+1}}\, P^{2\to r\,0} \right) \\
&\equiv \quad P^{1\to r+1} \left(Y_{\underline{2\to r+1}} \right)^{\langle 2\ldots r\,0}_{2\ldots r+1\rangle}\ ,
\end{aligned}
$$

$$
\begin{aligned}
& P_{1\to r+1} \left((-1)^{r}\, f^{(R)}_{\underline{1\to r+1}}\, R_{\underline{r}} \cdots R_{\underline{1}} - \overline{f}^{(R)}_{\underline{1\to r+1}} \right) P^{1\to r+1} \\
& = \left((-1)^{r}\, f_{1\to r+1}\, \sigma_r \sigma_{r-1} \cdots \sigma_1 - \overline{f}_{1\to r+1} \right)\ ,
\end{aligned}
$$

which can be obtained directly using the explicit form of the R-matrix (2).

The equation (34) is fulfilled in view of the identity (27) which has been proven in Lemma 1. Indeed, if in the identity (27) we take the R-matrix representation for the braid group $\rho_R(B_{M+1})$: $\rho_R(\hat{\sigma}_k) = R_{\underline{k}}$ and act on this identity from the left and right by the projectors $P_{1\to r+1}$, $P^{1\to r\,0}$, then we deduce (34). This completes the proof of the Proposition 2. •

In the next Section we need more information about the R-matrix representation of the braid group just considered. Introduce the Jucys-Murphy elements $J_r \in \rho_R(B_{M+1})$:

$$
J_1 = 1\ , \quad J_{r+1} = R_{\underline{r}}\, J_r\, R_{\underline{r}}\ . \tag{35}
$$

These elements form a commuting set in $\rho_R(B_{M+1})$: $[J_r,\, J_m] = 0$ and satisfy

$$
R_{\underline{m}}\, J_r = J_r\, R_{\underline{m}}\ , \quad m < r\ . \tag{36}
$$

Proposition 3. *The explicit form of the components*

$$
Z_{r+1} := P_{1\to r+1}(J_{r+1}) P^{1\to r\,0}
$$

of Jucys-Murphy elements (35) in terms of the structure constants C_n and σ_m is:

$$
\begin{gathered}
Z_{r+1} = C_r + \sigma_r\, C_{r-1}\, \delta_{r+1} + \sigma_r\, \sigma_{r-1}\, C_{r-2}\, \delta_r\, \delta_{r+1} + \ldots \\
+\sigma_r\, \sigma_{r-1} \ldots \sigma_2\, C_1\, \delta_3 \ldots \delta_{r+1} \;.
\end{gathered} \tag{37}
$$

The elements Z_r have the following properties

$$
\sigma_m\, Z_r = Z_r\, \sigma_m \;, \quad m \leq r-2 \;, \tag{38}
$$

$$
Z_{r+1} = C_r + \sigma_r\, Z_r\, \delta_{r+1} \;. \tag{39}
$$

Proof. Using the definition of Jucys-Murphy elements (35) and the representation (2) one can rewrite $P_{1\to r+1}(J_{r+1}) P^{1\to r\, 0}$ in the form

$$
\begin{gathered}
P_{1\to r+1}\, (R_{\underline{r}} \ldots (R_{\underline{1}})^2 \ldots R_{\underline{r}})\, P^{1\to r\, 0} \\
= P_{1\to r+1}\, \left(R_{\underline{r}}\, R_{\underline{r-1}} \ldots R_{\underline{1}} \right) P^{0\; 2\to r+1}\, \delta_2 \delta_3 \ldots \delta_{r+1} \;.
\end{gathered}
$$

Then, again using (2) we deduce (37). To obtain the property (38) apply the operators $P_{1\to r}$ and $P^{1\to r-1\, 0}$ to (36) from the left and from the right (for $m \leq r-2$). The recurrent relation (39) is a direct consequence of the relations (37). •

5 Relation to bar chain complex

We recall first some standard material. For every associative algebra $\mathcal{U}$ one forms the bar complex $(C_n(\mathcal{U}), b)$:

a) $C_n(\mathcal{U})$ is the space $\underbrace{\mathcal{U} \otimes \ldots \otimes \mathcal{U}}_{n}$;

b) for

$$
T = a_1 \otimes a_2 \otimes \ldots \otimes a_n \in C_n(\mathcal{U}) \;, \tag{40}
$$

the action of the boundary operator $b : C_{n+1}(\mathcal{U}) \to C_n(\mathcal{U})$ is given by

$$
b\,(a_1 \otimes a_2 \otimes \ldots \otimes a_{n+1}) = \sum_{i=1}^{n} (-1)^{n-i}\, (a_1 \otimes \ldots \otimes a_i\, a_{i+1} \otimes \ldots \otimes a_{n+1}) \;. \tag{41}
$$

Using the associativity of $\mathcal{U}$ one can check directly that $b^2\,(a_1 \otimes a_2 \otimes \ldots \otimes a_{n+1}) = 0$. Moreover, for a unital algebra $\mathcal{U}$, the complex $(C_n(\mathcal{U}), b)$ is acyclic since one can define a homotopy operator δ: $C_n(\mathcal{U}) \to C_{n+1}(\mathcal{U})$

$$\delta(a_1 \otimes a_2 \otimes \ldots \otimes a_n) = (a_1 \otimes a_2 \otimes \ldots \otimes a_n \otimes 1)$$

which satisfies $\delta\, b + b\, \delta = id$.

Let $\mathcal{U}$ be the q-Lie algebra with generators χ_i and defining relations (5). Consider the chains T (as in (40)) with all elements a_k belonging to the set $\{\chi_j\}$ of the generators of the q-Lie algebra and define their q-antisymmetric linear combination which is given by the contraction with the numerical tensor $A_{1\to n} \in End(V^{\otimes n})$ (14):

$$\chi_{1\rangle} \wedge \chi_{2\rangle} \wedge \ldots \wedge \chi_{n\rangle} := A_{1\to n} \, \{\chi_{1\rangle} \otimes \chi_{2\rangle} \otimes \ldots \otimes \chi_{n\rangle}\} \in C_n(\mathcal{U}) \; . \tag{42}$$

In fact we shall need the following chains:

$$a \otimes \{\chi_{1\rangle} \wedge \chi_{2\rangle} \wedge \ldots \wedge \chi_{n\rangle}\} \in C_{n+1}(\mathcal{U}) \tag{43}$$

with arbitrary $a \in \mathcal{U}$. Denote the subspace of such chains by $C_{n+1}(\mathcal{U}, \wedge\mathcal{L}) \in \mathcal{U} \otimes \underbrace{\mathcal{L} \wedge \ldots \wedge \mathcal{L}}_{n}$, where $\mathcal{L}$ is a linear span of $\{\chi_i\}$.

Now we formulate one of the main results of this paper.

Proposition 4. *The boundary operator b (41) preserves the q-antisymmetry,*

$$b: \quad C_{n+1}(\mathcal{U}, \wedge\mathcal{L}) \longrightarrow C_n(\mathcal{U}, \wedge\mathcal{L}) \; . \tag{44}$$

In other words, we have a complex $(C_n(\mathcal{U}, \wedge\mathcal{L}), b)$ which is a subcomplex of the bar complex $(C_n(\mathcal{U}), b)$.

Proof. Acting by the boundary operator (41) on the q-antisymmetric chains (43), we deduce

$$b\,\{a \otimes \chi_{1\rangle} \wedge \chi_{2\rangle} \wedge \ldots \wedge \chi_{n+1\rangle}\} = (-1)^n \, A_{1\to n+1} \, \{a\, \chi_{1\rangle} \otimes \chi_{2\rangle} \otimes \ldots \otimes \chi_{n+1\rangle}\}$$

$$+ a \otimes A_{1\to n+1} \, \left[\sum_{k=1}^{n} (-1)^{n-k} \, \{\chi_{1\rangle} \otimes \ldots \otimes \chi_{k\rangle}\, \chi_{k+1\rangle} \otimes \ldots \otimes \chi_{n+1\rangle}\} \right] \tag{45}$$

$$= (-1)^n \, \overline{f}_{1\to n+1} \, \{a \, \chi_{1\rangle} \otimes \chi_{2\rangle} \wedge \ldots \wedge \chi_{n+1\rangle}\}$$

$$+a \otimes A_{1\to n+1} \sum_{k=1}^{n} (-1)^{n-k} \, t^{\langle k}_{k,k+1\rangle} \{\chi_{1\rangle} \otimes \ldots \otimes \chi_{k\rangle} \otimes \chi_{k+2\rangle} \otimes \ldots \otimes \chi_{n+1\rangle}\} \; , \tag{46}$$

where structure constants t^i_{jk} satisfy

$$C^{\langle k}_{kk+1\rangle} = (1 - \sigma_k) \, t^{\langle k}_{kk+1\rangle} \; , \tag{47}$$

tensors $\overline{f}_{1\to n+1}$ are introduced in (14) – (16) and we have used the defining relations (5) and the properties of "antisymmetrizers"

$$A_{1\to n+1} \chi_{k\rangle} \, \chi_{k+1\rangle} = A_{1\to n+1} t^{\langle k}_{kk+1\rangle} \, \chi_{k\rangle} \; .$$

Thus, to prove (44) we need to represent the second term in (46) in the form

$$a \otimes W^{\langle 1\ldots n}_{1\ldots n+1\rangle} \, \{\chi_{1\rangle} \wedge \ldots \wedge \chi_{n\rangle}\} = a \otimes W^{\langle 1\ldots n}_{1\ldots n+1\rangle} \, A_{1\to n} \, \{\chi_{1\rangle} \otimes \ldots \otimes \chi_{n\rangle}\} \; . \tag{48}$$

Therefore it is sufficient to prove:

Lemma 2. *There exist tensors $W_{n+1} := W^{\langle 1\ldots n}_{1\ldots n+1\rangle}$ such that*

$$A_{1\to n+1} \left(\sum_{k=1}^{n} (-1)^{n-k} \, t^{\langle k}_{k,k+1\rangle} \, \delta_{k+2\to n+1} \right) = W_{n+1} \, A_{1\to n} \tag{49}$$

where $\delta_{n+1\to n} := 1$, $\delta_{k+2\to n+1} := \delta_{k+2} \cdots \delta_{n+1}$ is a shift operator $i \to i+1$ for $k < i \leq n$. The explicit form of the tensors W_n (in terms of the quantum Lie algebra structure constants σ^{ij}_{kl} and C^i_{jk}) is:

$$\begin{aligned} W_{n+1} &= C_n - (1 - \sigma_n) \, C_{n-1} \, \delta_{n+1} + (1 - \sigma_{n-1} + \sigma_n \, \sigma_{n-1}) \, C_{n-2} \, \delta_n \, \delta_{n+1} - \ldots \\ &\ldots + (-1)^{n+1} \, [1 - \sigma_2 + \sigma_3 \, \sigma_2 - \ldots + (-1)^{n+1} \sigma_n \cdots \sigma_2] \, C_1 \, \delta_3 \cdots \delta_{n+1} \\ &= Z_{n+1} - Z_n \, \delta_{n+1} + Z_{n-1} \, \delta_n \, \delta_{n+1} + \ldots + (-1)^{n+1} \, Z_2 \, \delta_3 \ldots \delta_{n+1} \; , \end{aligned} \tag{50}$$

where the elements Z_k are defined in (37), (39) (we fix $Z_2 = C_1 = W_2$).

Proof. To prove (49) we use the identities (7) – (10) for the structure constants and the relation (47) between C^j_{ik} and t^j_{ik}.

Eq. (49) is obviously fulfilled for $n = 1$. Assume that (49) is correct for $n = m - 1$. We have to prove (49) for $n = m$. For the left hand side of (49) we obtain:

$$A_{1\to m+1} \left(\sum_{k=1}^{m} (-1)^{m-k} \, t^{\langle k}_{k,k+1\rangle} \, \delta_{k+2} \cdots \delta_{m+1} \right)$$

$$= A_{1\to m+1} \left(\left[\sum_{k=1}^{m-1} (-1)^{m-k} \, t^{\langle k}_{k,k+1\rangle} \, \delta_{k+2} \ldots \delta_m \right] \delta_{m+1} + t^{\langle m}_{m,m+1\rangle} \right)$$

$$= \left(-f_{m+1} \, W_m \, \delta_{m+1} + f_{m+1} \, f_m \, t^{\langle m}_{m,m+1\rangle} \right) \, A_{1\to m-1}$$

$$= \left[-f_{m+1} \, W_m \, \delta_{m+1} + x_{m+1} \, C_m \right] \, A_{1\to m-1} \ .$$

In the second equality we used (14), (49); in the third equality we used (19) with $n - m = 2$. Here and below, to simplify notation, we put $f_m := f_{1\to m}$ and $x_m := x^{(2)}_m$.

Comparing with the right-hand side of (49), we obtain the equation

$$-f_{m+1} \, W_m \, \delta_{m+1} + x_{m+1} \, C_m = W_{m+1} \, f_m \ , \tag{51}$$

which is an identity for $m = 2$ in view of (9) and (10). We prove (51) by induction. For this we note (formula (50)) that one has the inductive relation

$$W_{n+1} = Z_{n+1} - W_n \, \delta_{n+1} \ . \tag{52}$$

Substitute to the left-hand side of (51) the recurrence relations (15), (23), (52) and take into account relations (9), (39), (38) and the base of induction:

$$-W_m \, \delta_{m+1} + f_m \, \sigma_m \, (Z_m - W_{m-1} \, \delta_m) \, \delta_{m+1} + (f_m + x_m \, \sigma_m \sigma_{m-1}) \, C_m$$

$$= -W_m \, \delta_{m+1} + f_m \, (\sigma_m \, Z_m \, \delta_{m+1} + C_m) + W_m \, f_{m-1} \, \delta_{m+1} \, \sigma_{m-1}$$

$$= (Z_{m+1} - W_m \, \delta_{m+1}) - f_{m-1} \, \sigma_{m-1} \, Z_{m+1} + W_m \, \delta_{m+1} \, f_{m-1} \, \sigma_{m-1}$$

$$= (Z_{m+1} - W_m \, \delta_{m+1})(1 - f_{m-1} \, \sigma_{m-1}) = W_{m+1} \, f_m \ .$$

Thus, we proved (51) and, therefore, (49). •
This finishes the proof of the Proposition 4. •

The identity (49) guarantees that the q-antisymmetric chains (43) form the subcomplex in the bar complex for q-Lie algebra $\mathcal{U}$ (5). The identity

$$(b)^2 \, \{ a \otimes \chi_{1\rangle} \wedge \chi_{2\rangle} \wedge \ldots \wedge \chi_{n+1\rangle} \} = 0$$

for the boundary operator (45) directly follows from the identity $b^2 = 0$ for the bar differential b (41).

In analogy with the theory of usual Lie algebras (see e. g. [8]), we relate the complex $(C_n(\mathcal{U}, \wedge\mathcal{L}), b)$ with the standard complex for the quantum Lie algebra. Namely, we relate the chains

$$C_n = a \otimes A_{1\to n}(\chi_{1\rangle} \otimes \ldots \otimes \chi_n) \in C_n(\mathcal{U}, \wedge\mathcal{L})$$

with wedge polynomials in γ's with coefficients in $\mathcal{U}$

$$M_n = a\, \gamma_{1\rangle} \wedge \gamma_{2\rangle} \wedge \ldots \wedge \gamma_{n\rangle} \ , \quad a \in \mathcal{U} \ , \tag{53}$$

by means of the one-to-one map $\mathbf{i}: \ C_n \leftrightarrow M_n$,

$$\mathbf{i}(a\,\{\gamma_{1\rangle} \wedge \gamma_{2\rangle} \wedge \ldots \wedge \gamma_{n\rangle}\}) = a \otimes A_{1\to n}\,\{\chi_{1\rangle} \otimes \chi_{2\rangle} \otimes \ldots \otimes \chi_{n\rangle}\} \tag{54}$$

The BRST operator Q (24), (25), (32) acts on the expressions (53) from the right and, according to (13), the variables Ω's in Q should be considered as "right q-derivatives" over γ's (in particular $\forall a \in \mathcal{U}$ the right action of Ω^i on a gives zero).

Proposition 5.

1. $Q^2 = 0$;
2. *The map* $\mathbf{i}$ *(54) is the chain map,* $b \circ \mathbf{i} = \mathbf{i} \circ Q$.

Proof. Consider only first two terms in the expression (24):

$$Q = \Omega^{\langle 1}\, \chi_{1\rangle} - \Omega^{\langle 2} \wedge \Omega^{\langle 1}\, t^{\langle 1}_{12\rangle}\, \gamma_{1\rangle} + \ldots \ . \tag{55}$$

Using (13), (17), (37) and (39) one can prove by induction the following relations:

$$\gamma_{1\rangle} \wedge \ldots \wedge \gamma_{n\rangle}\, \Omega^{\langle n} = (-1)^n\, \Omega^{\langle 0}\, \sigma_0^{-1} \ldots \sigma_{n-1}^{-1}\, \gamma_{0\rangle} \wedge \ldots \wedge \gamma_{n-1\rangle}$$
$$+ (-1)^{n-1}\, \overline{f}^{(\sigma)}_{1\to n}\, \sigma_1^{-1} \ldots \sigma_{n-1}^{-1}\, \gamma_{1\rangle} \wedge \ldots \wedge \gamma_{n-1\rangle} \ , \tag{56}$$
$$\gamma_{1\rangle} \wedge \cdots \wedge \gamma_{n-1\rangle}\, \chi_{n\rangle} = \sigma_{n-1} \ldots \sigma_1\, \chi_{1\rangle} \gamma_{2\rangle} \wedge \cdots \wedge \gamma_{n\rangle} + Z_n\, \gamma_{1\rangle} \wedge \cdots \wedge \gamma_{n-1\rangle} \ .$$

Thus, using (20) and (56), the right action of the first two terms (55) of Q on (53) gives

$$Q\left(a\,\gamma_{1\rangle}\wedge\ldots\wedge\gamma_{n\rangle}\right)=(-1)^{n-1}\,\overline{f}^{(\sigma)}_{1\to n}\,\left(a\,\chi_{1\rangle}\gamma_{2\rangle}\wedge\ldots\wedge\gamma_{n\rangle}\right)+$$

$$+(-1)^{n-1}\overline{f}^{(\sigma)}_{1\to n}\,\sigma_1^{-1}\ldots\sigma_{n-1}^{-1}\,Z_n\,\left(a\,\gamma_{1\rangle}\wedge\ldots\wedge\gamma_{n-1\rangle}\right)-$$

$$-x_n^{(n-2)}\,(\sigma_1^{-1}\ldots\sigma_{n-1}^{-1})\,(\sigma_1^{-1}\ldots\sigma_{n-2}^{-1})\,C_{n-1}\,\left(a\,\gamma_{1\rangle}\wedge\ldots\wedge\gamma_{n-1\rangle}\right)+\ldots\,. \tag{57}$$

According to the Proposition 1 the first two terms in (55), together with the condition

$$Q^2|_{\text{terms linear in }\chi}=0\;, \tag{58}$$

define BRST operator Q uniquely. So we need only to check that the formula (57) (for $n=1,2$) is compatible with $b\circ\mathbf{i}=\mathbf{i}\circ Q$ and eqs. (46), (48), (54). It can be done directly. Therefore $Q^2=0$ and $\mathbf{i}$ is a chain map. •

The fact that the map $\mathbf{i}$ (54) is the chain map demonstrates the relation between the standard complex for the q-Lie algebra (with our BRST operator) and the subcomplex $(C_n(\mathcal{U},\wedge\mathcal{L}),b)$ of the bar complex $(C_n(\mathcal{U}),b)$.

Remark. One can show that

$$W_{n+1}=P_{1\to n+1}\,(\overline{f}^{(R)}_{\underline{1\to n+1}}-1)\,R_{\underline{1}}\ldots R_{\underline{n}}\,P^{1\to n,0}\;.$$

The relation (51) can be obtained from the identity

$$f^{(R)}_{\underline{1\to m+1}}\,(\overline{f}^{(R)}_{\underline{1\to m}}\,R_{\underline{1}}\ldots R_{\underline{m-1}})R_{\underline{m}}=(\overline{f}^{(R)}_{\underline{1\to m+1}}\,R_{\underline{1}}\ldots R_{\underline{m}})\,f^{(R)}_{\underline{1\to m}} \tag{59}$$

by action of $P^{1\to m,0}$ and $P_{1\to m+1}$ from the right and left. The identity (59) is equivalent to

$$f^{(R)}_{\underline{1\to m+1}}\,\overline{f}^{(R)}_{\underline{1\to m}}=\overline{f}^{(R)}_{\underline{1\to m+1}}\,f^{(R)}_{\underline{2\to m+1}}$$

which is nothing but the associativity condition $x^{(1)}_{m+1}\,x^{(m-1)}_{m}=x^{(m)}_{m+1}T_1x^{(1)}_mT_1^{-1}$ considered in the Remark in Section 3. To prove all these statements we need the following relations

$$(\overline{f}^{(R)}_{\underline{1\to n+1}}-1)\,R_{\underline{1}}\ldots R_{\underline{n}}\,P^{1\to n,0}$$

$$=P^{1\to n+1}\,W_{n+1}+\textstyle\sum_{k=1}^{n}(-1)^k\,P^{1\to k}(1-P^{k+1})\,P^{k+2\to n+1}\delta_{k+2}\ldots\delta_{n+1}$$

and

$$f^{(R)}_{\underline{1\to m}}\, P^{1\to m} = P^{1\to m}\, f^{(\sigma)}_{1\to m} \ .$$

6 A generalization

6.1. We consider the similarity transformation of the R-matrix

$$R_{\underline{12}} \to R'_{\underline{12}} = \mathcal{D}_{\underline{1}}\, \mathcal{D}_{\underline{2}}\, R_{\underline{12}}\, \mathcal{D}^{-1}_{\underline{1}}\, \mathcal{D}^{-1}_{\underline{2}} \tag{60}$$

which corresponds to the linear transformation of the generators $L_B \to \mathcal{D}^C_B\, L_C$. The new matrix (60) satisfies Yang-Baxter equation (6) for all numerical matrices $\mathcal{D}$. If the matrix $\mathcal{D}$ has the special form

$$\mathcal{D}^0_0 = 1 \ , \quad \mathcal{D}^0_i = \mathcal{D}^i_0 = 0 \ , \quad \mathcal{D}^i_j = D^i_j \ ,$$

then the new R matrix (60) has the same form (1) with transformed structure constants

$$\sigma_{12} \to \sigma'_{12} = D_1\, D_2\, \sigma_{12}\, D^{-1}_1\, D^{-1}_2 \ , \quad C^{\langle 2|}_{|12\rangle} \to C\,'^{\langle 2|}_{|12\rangle} = D_1\, D_2\, C^{\langle 2|}_{|12\rangle}\, D^{-1}_2 \ . \tag{61}$$

This transformation leaves invariant the linear space (denoted by $\mathcal{L}$) spanned by the N elements χ_i. The q-Lie algebra is quadratic for any choice of the basis in $\mathcal{L}$ but the structure constants in (5) change according to (61).

The interesting special case is when the matrix R is conserved under transformation (60) $R_{\underline{12}} = R'_{\underline{12}}$. Then the $N \times N$ matrix D^i_j is such that

$$D_1\, D_2\, \sigma_{12} = \sigma_{12}\, D_1\, D_2 \ , \quad D_1\, D_2\, C^{\langle 2|}_{|12\rangle} = C^{\langle 2|}_{|12\rangle}\, D_2 \ . \tag{62}$$

Lemma. *In this case the R-matrix $\mathcal{D}_{\underline{1}}\, R_{\underline{12}}\, \mathcal{D}^{-1}_{\underline{1}}$ satisfies Yang-Baxter equation.*

The proof is straightforward. Now one can consider a quadratic algebra with defining relations

$$\left(\mathcal{D}_{\underline{1}}\, R_{\underline{12}}\, \mathcal{D}^{-1}_{\underline{1}}\right)\, L'_{\underline{1}\rangle}\, L'_{\underline{2}\rangle} = L'_{\underline{1}\rangle}\, L'_{\underline{2}\rangle} \ .$$

In the components, for the quantum vector $L'_A = \{\xi_0, \xi_i\}$, we obtain

$$\xi_i\, \xi_0 = D^j_i\, \xi_0\, \xi_j \ \text{ and } \ (1 - D_1\, \sigma_{12}\, D^{-1}_1)\, \xi_{1\rangle}\, \xi_{2\rangle} = D_1\, C^{\langle 1}_{12\rangle}\, \xi_0\, \xi_{1\rangle} \ . \tag{63}$$

This algebra is related to the q-Lie algebra (5) by a transformation

$$\chi_i = \xi_0^{-1} \xi_i \ . \tag{64}$$

6.2. There is a generalization of the exterior algebra (11) –(13) and (17) arising from the considerations just above. One can introduce a grading operator g with commutation relations

$$g \, \gamma_{1\rangle} = D_1 \, \gamma_{1\rangle} \, g \ , \quad g \, \Omega^{\langle 1} = \Omega^{\langle 1} \, D_1^{-1} \, g \ , \quad g \, \chi_{1\rangle} = D_1 \, \chi_{1\rangle} \, g \ , \tag{65}$$

where D^i_j is the matrix which satisfies (62). Note that the grading operator g can be included into play if we consider the quadratic algebra (63) and take (64) and $g = \xi_0^{-1}$.

Relations (65) are consistent with relations (5), (11) –(13) and (17). Then the Woronowicz differential algebra (11) –(13), (17) can be rewritten for the new generators

$$\tilde{\Omega}^{\langle 1} = \Omega^{\langle 1} \, g^{-1} \ , \quad \tilde{\gamma}_{1\rangle} = g \, \gamma_{1\rangle}$$

in the form:

$$\tilde{\gamma}_{2\rangle} \, \tilde{\Omega}^{\langle 2} = - \tilde{\Omega}^{\langle 1} \, D_1^{-1} \, \sigma_{12}^{-1} \, D_1 \, \tilde{\gamma}_{1\rangle} + I_2 \ , \tag{66}$$

$$\tilde{\Omega}^{\langle 1} \wedge \tilde{\Omega}^{\langle 2} \wedge \ldots \wedge \tilde{\Omega}^{\langle n} = \tilde{\Omega}^{\langle 1} \otimes \tilde{\Omega}^{\langle 2} \otimes \ldots \otimes \tilde{\Omega}^{\langle n} \, A^{(D)}_{1\to n} \ , \tag{67}$$

$$\tilde{\gamma}_{1\rangle} \wedge \tilde{\gamma}_{2\rangle} \ldots \wedge \tilde{\gamma}_{n\rangle} = A^{(D)}_{1\to n} \, \tilde{\gamma}_{1\rangle} \otimes \tilde{\gamma}_{2\rangle} \ldots \otimes \tilde{\gamma}_{n\rangle} \ , \tag{68}$$

$$\chi_{2\rangle} \, \tilde{\Omega}^{\langle 2} = \tilde{\Omega}^{\langle 1} \left(\sigma_{12} D_1 \, \chi_{1\rangle} + C^{\langle 2}_{12\rangle} \right) \ , \quad \tilde{\gamma}_{1\rangle} \, \chi_{2\rangle} = \sigma_{12} D_1 \, \chi_{1\rangle} \, \tilde{\gamma}_{2\rangle} + C^{\langle 1}_{12\rangle} \, \tilde{\gamma}_{|1\rangle} \ , \tag{69}$$

where $A^{(D)}_{1\to n} = D_1^{-1} \cdots D_{n-1}^{-1} \, A_{1\to n} \, D_1 \cdots D_{n-1}$.

This generalized algebra embeds into the bar complex; the embedding is similar to the one in the particular case $D = 1$ and we don't give details.

6.3. The quantum Lie algebras defined by eqs (5), (7)-(10) generalize the usual Lie (super)algebras. Indeed, in the non-deformed case, the braid matrix $\sigma^{mk}_{ij} = (-1)^{(m)(k)} \, \delta^m_j \, \delta^k_i$ is a super-permutation matrix (here $(m) = 0, 1$ is the parity of a generator χ_m). Eqn. (7) is fulfilled (and we have additionally $\sigma^2 = 1$). Eqn. (8) coincides with the Jacobi identity for Lie (super)-algebras. Eqn. (9) is then equivalent to the Z_2-homogeneity condition $C^i_{jk} = 0$ for $(i) \neq (j) + (k)$. Eqn. (10) follows from (9) in this case.

In this sense the exterior algebras (11) –(13), (66) – (68) generalize the Heisenberg algebras of the fermionic (and bosonic) ghosts and anti-ghosts. The grading matrix D now is the matrix $D_n^m = (-1)^{(n)} \delta_n^m$ which defines the well known automorphism of superalgebras, and we have

$$(D_1 \sigma_{12} D_1^{-1})_{ij}^{mk} = -(-1)^{((m)+1)((k)+1)} \, \delta_j^m \, \delta_i^k \; .$$

7 Conclusion

We have investigated the special class of quadratic algebras $\mathcal{U}$, the quantum Lie algebras (5). We have considered the exterior extension of $\mathcal{U}$ by the ghost algebra (11), (12), (13), (17) and constructed the BRST operator Q. Using the BRST operator (24), (25), (32) one can build analogues of the standard and de Rham complexes for quantum Lie algebras. As we have shown in this paper, there is a map $\mathbf{i}$ from the standard complex to the subcomplex in the bar complex of $\mathcal{U}$. The map $\mathbf{i}$ induces (from the bar differential b) a differential on the standard complex. We compared this induced differential with the operator Q. The operator Q is defined uniquely by two properties: the initial terms (55) and the condition (58). We verified that the induced differential satisfies these two properties. Therefore it coincides with Q. Moreover it follows that $Q^2 = 0$. For the construction of the map $\mathbf{i}$ we need to check a number of complicated identities on the structure constants σ_{ij}^{kl} and C_{ij}^k. The similar identities have appeared in the inductive definition of the BRST operator for q-Lie algebras. An elegant proof of some of these identities is based on combining the constants σ_{ij}^{kl} and C_{ij}^k into a bigger matrix R_{CD}^{AB} which realizes an R-matrix representation of the braid group $\mathcal{B}_\bullet$. Then we use the properties of the Jucys - Murphy elements in $\mathcal{B}_\bullet$, the quantum shuffle product and several important identities in the braid group algebra.

We presented a generalization of the constructions above to the situation when a q-Lie algebra is equipped with a grading operator.

Some results from Section 5 can be considered as an explicit realization of certain facts from the paper [3] for the special choice of the inhomogeneous quadratic algebras.

Note that the particular example of our construction of the BRST operator for the case of the quantum Lie algebra $U_q(gl(N))$ has been already considered in detail in our paper [14]. In this case we have constructed also

the quantum analogues of the anti-BRST operator and the quantum Laplace operator.

Also,we hope that our construction of the BRST operator for the quantum Lie algebras will be useful for constructions of BRST operators for any quadratic algebras (even for infinite dimensional algebras, such as W_3) and for producing the quantum W-algebras with the help of the quantum affine algebras (see [15] and references therein) by means of the quantum analogue of the Hamiltonian reduction procedure via the quantum BRST technique.

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